Semisimilar Solutions to Unsteady Boundary-Layer Flows Including Separation

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A study is made of a certain class of flows for which semisimilar solutions to the unsteady two-dimensional laminar boundary-layer equations may be obtained. It is shown that such solutions are possible when the velocity at the edge of the boundary layer, $u_\delta(x, t)$, is an arbitrary function of the variable ξ where ξ is either $x/(1-Bt)$ or $(x+Kt)/(1-Bt)$. If the constant K is choosen properly the latter case represents the same external flow as the former case but observed in a reference system moving with the unsteady separation point. It is possible, therefore, to study the unsteady separation phenomenon in this reference system where separation is most easily identified and analyzed. Numerical solutions have been obtained for an unsteady linearly retarded flow where $u_\delta(x,t) = U_\infty - A\xi$. These solutions verify the Moore-Rott-Sears model for unsteady boundary-layer separation in which separation appears as a point of vanishing shear and velocity, within the boundary layer, in a reference system moving with separation. Solutions such as those presented here may also be used as test cases for the new numerical techniques where unsteady boundary layers are analyzed using three-dimensional finite difference techniques.

Nomenclature

A	= arbitrary constant
a, b, c, d, e, h	= arbitrary functions defined by Eqs. (8)
B	= arbitrary constant
f	= nondimensional stream function
f_w''	= nondimensional wall shear
g	= arbitrary function of x and t [see Eq. (3)]
K	= arbitrary constant
$m(t)$	= arbitrary function of time
$p(t)$	= arbitrary function of time
t	= time
U_∞	= freestream velocity
U_s	= velocity of the separation point
u	= x component of boundary-layer velocity
u^*	= u/U_∞
$\bar{u}^*$	= $\bar{u}/U_\infty$
u_δ	= inviscid velocity exterior to the boundary layer
v	= y component of boundary-layer velocity
x	= physical coordinate in the flow direction
x^*	= Ax
$\bar{x}^*$	= $A\bar{x}$
x_{sep}	= physical location of the separation point
y	= physical coordinate normal to the flow direction
Δ^*	= pseudo-displacement thickness
η	= scaled y coordinate
λ	= B/AU_∞
ν	= kinematic viscosity
ξ	= scaled streamwise coordinate, function of x and t
ξ^*	= $A\xi$
ξ_0	= streamwise location of the separation point in the (ξ, η) coordinate system
ξ_0^*	= $A\xi_0$
ψ	= stream function

Superscript

"‾"	= overbars denote quantities in the moving coordinate system

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Introduction

THE solution of many practical fluid mechanics problems hinges on the understanding of the behavior of the unsteady boundary layer. Problems ranging from the dynamic stall of helicopter rotor blades to the propulsion of fish are strongly influenced by such boundary layers. The theory of unsteady boundary layers, is, however, rather poorly developed as compared to the theory of steady boundary layers. The reason for this state of affairs is two fold. First, the introduction of another independent variable, time, increases appreciably the mathematical complexity of the problem. Second, until quite recently the phenomenon of unsteady boundary-layer separation was rather poorly understood.

The additional mathematical difficulty associated with the unsteady boundary-layer problem has lead several investigators (e.g., Refs. 1 and 2) to explore numerical methods for the calculation of these boundary layers. The solutions available against which these numerical techniques may be tested are limited in number and are for very special flows. There is a need for additional solutions, particularly for problems in which separation is approached, which may be used to test the numerical techniques.

A more serious difficulty is that associated with our lack of knowledge regarding the phenomenon of unsteady separation. In many of the earlier analyses of unsteady boundary layers the criterion for separation was taken to be the same as that for steady flow, i.e., the vanishing of the shear at the body surface. Moore,[3] Rott,[4] and Sears[5] all noted that the vanishing of the wall shear held no special significance in unsteady boundary layers and developed, independently, a model for unsteady separation which is presently known as the Moore-Rott-Sears model. This model postulates that the unsteady separation point is characterized by the simultaneous vanishing of the shear and the velocity, at some point away from the wall, *as seen by an observer moving with separation.* The difficulty in applying this model in practice lies in the fact that the moving reference frame (moving with separation) is not, in general, known a priori. Moore[3] and later Telionis[6] have also pointed out that the boundary-layer equations have a singularity of the Goldstein type at the point of separation as described in the Moore-Rott-Sears model.

Since the development of this model there have been two attempts to verify it, both using numerical techniques. Telionis and Werle[7] studied the steady flow over an idealized moving wall and pointed out that separation in this case was characterized by the simultaneous vanishing of the shear and velocity in a singular manner within the boundary layer. Since there is an analogy between certain unsteady boundary-layer problems and steady moving-wall problems, they concluded that the Moore-Rott-Sears model with a separation singularity is the correct description of unsteady separation. More recently, Telionis, Tsahalis, and Werle[8] employed an implicit finite-difference scheme in which there were three variables (x, y, t) to study an unsteady form of the linearly retarded flow of Howarth. They observed that the y component of velocity v within the boundary layer increases in a singular manner as separation is approached, thus verifying the postulated singular behavior of the unsteady boundary-layer equations near separation.

In connection with this latter study it is important to mention that since vanishing shear at the wall does not signify separation, solution of the unsteady boundary-layer equations often leads to integration into a region where at least part of the flow is reversed. In this case one encounters the problem of integrating parabolic equations in the "upstream" direction, a procedure which may lead to instabilities.

The problem of unsteady separation is often linked, intuitively, with the problem of steady flow over a moving wall. Ludwig[9] has made an experimental study of steady separation over a moving wall and shown that separation in this case has the physical characteristics postulated for the unsteady separation point. The connection between unsteady separation and separation over a moving wall has, however, remained illusive. In the present work, a connection is made between unsteady separation and separation over a moving wall.

With regard to the question of unsteady separation, one must note that at ordinary Reynolds numbers, laminar separation represents the end of the region which is unaffected by downstream influences. As a result, the turbulence in the wake may perturb the separation point leading to an additional unsteadiness in the separation point. These effects are probably of higher order and are not considered here.

It is clear that there is a need for additional solutions of the unsteady boundary-layer equations which may be used as test cases for numerical integration techniques and for solutions which will yield additional understanding of the unsteady separation phenomenon. The present paper attempts to satisfy both these needs. It is shown that there is a rather general class of unsteady boundary-layer problems which may be treated by the method of semisimilar solutions. In this set of problems the unsteady problem in three independent variables is transformed into an equivalent problem in two independent variables. When this is accomplished, standard numerical techniques for solving nonsimilar, steady, two-dimensional boundary-layer equations may be used with considerable success. In addition, it is shown that for this class of problems it is a rather simple matter to transform the problem into a coordinate system which moves with the unsteady separation point. In this coordinate system unsteady separation is easily identified and the problem of integrating upstream is avoided.

The solutions obtained verify the Moore-Rott-Sears model for unsteady separation and the singularity postulated by Moore and Telionis and provide solutions to the unsteady boundary layer equations which may be used as test cases to verify three dimensional numerical techniques.

Analysis

For an incompressible fluid the unsteady two-dimensional boundary-layer equations are

$$(\partial u/\partial x)+(\partial v/\partial y)=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial u_\delta}{\partial t}+u_\delta\frac{\partial u_\delta}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

with the boundary conditions

$$u(x,0,t)=v(x,0,t)=0, \quad \lim_{y\to\infty} u(x,y,t)=u_\delta(x,t)$$

where $u_\delta(x, t)$ is the known unsteady velocity distribution outside the boundary layer. We wish to obtain solutions to these equations which have physical meaning and which yield some insight into the nature of the unsteady separation phenomenon. The approach used here is that of semisimilar solutions in which the number of independent variables is reduced from three to two. Introduction of the new variables

$$\eta = y/g(x,t)\nu^{1/2}, \qquad \xi=\xi(x,t) \tag{3}$$

where $g(x, t)$ and $\xi(x, t)$ are as yet unknown functions, and of a dimensionless stream function $f(\xi, \eta)$ related to the stream function $\psi(x, y, t)$ by

$$f(\xi,\eta)=\psi(x,y,t)/\nu^{1/2}g(x,t)u_\delta(x,t) \tag{4}$$

yields this type of reduction.

The velocity components are given by

$$u=\partial\psi/\partial y=u_\delta(\partial f/\partial\eta) \tag{5}$$

$$v=-\frac{\partial\psi}{\partial x}=-\nu^{1/2}\left\{f\frac{\partial}{\partial x}(u_\delta g)+u_\delta g\left(\frac{\partial\xi}{\partial x}\frac{\partial f}{\partial\xi}-\frac{\eta}{g}\frac{\partial g}{\partial x}\frac{\partial f}{\partial\eta}\right)\right\} \tag{6}$$

The continuity equation is satisfied identically by the introduction of the stream function. If the velocity components and their derivatives are substituted into the momentum equation one obtains

$$\frac{\partial^3 f}{\partial\eta^3}+(d+e)f\frac{\partial^2 f}{\partial\eta^2}+d\left\{1-\left(\frac{\partial f}{\partial\eta}\right)^2\right\}+a\left(1-\frac{\partial f}{\partial\eta}\right)+\frac{b}{2}\eta\frac{\partial^2 f}{\partial\eta^2}-c\frac{\partial^2 f}{\partial\xi\,\partial\eta}+h\left\{\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2}-\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\xi\,\partial\eta}\right\}=0 \tag{7}$$

where

$$a=(g^2/u_\delta)(\partial u_\delta/\partial t) \qquad b=\partial g^2/\partial t \tag{8a, b}$$

$$c=g^2(\partial\xi/\partial t) \qquad d=g^2(\partial u_\delta/\partial x) \tag{8c, d}$$

$$2e=u_\delta(\partial g^2/\partial x) \qquad h=g^2u_\delta(\partial\xi/\partial x) \tag{8e, f}$$

If semisimilar solutions are to exist then a, b, c, d, e, and h must be functions of ξ alone. Three relations between these six functions are obtained by differentiating Eqs. (8a–c) with respect to x and using Eqs. (8d–f). These relations are

$$(a+b)d=2ae+cd'-a'h \tag{9a}$$

$$2ae=2ce'-hb' \tag{9b}$$

$$h(a+b+c')=c(h'+2e) \tag{9c}$$

where the primes here denote differentiation with respect to ξ.

Finally the boundary conditions written in terms of f and its derivatives are

$$f(\xi,0)=\frac{\partial f}{\partial\eta}(\xi,0)=0, \lim_{\eta\to\infty}\frac{\partial f}{\partial\eta}(\xi,\eta)=1 \tag{10}$$

The analysis to this point follows very closely that of Hayasi[10] who, as far as the authors have been able to determine, was the first to systematically study the problem of semisimilar solutions. Hayasi, however, limited his analysis to the cases where ξ was either equal to t (time dependent semisimilar) or equal to x (space-dependent semisimilar). A more interesting case is the general case where ξ is a function of both x and t.

The solution of the set of Eqs. (7, 8a–f, and 9a–c) is in general quite difficult. The six functions a, b, c, d, e, and h are related by the three Eqs. (9a–c). Thus three of these functions are arbitrary. Ideally, one would like to solve the problem directly for a given $u_\delta(x, t)$ by solving for the appropriate $g(x, t)$ and $\xi(x, t)$ from Eqs. (8a–f), subject to the constraints of Eqs. (9a–c). Once the functions a, b, c, d, e, and h are known the solution to Eq. (7), subject to the boundary conditions (10), is straightforward. This direct procedure is, however, usually not possible and one is forced to solve the problem indirectly. An indirect solution is possible if one assumes the form of any three of the functions a, b, c, d, e, and h. The remaining three functions are obtained by solving Eqs. (9). The external velocity $u_\delta(x, t)$ and the functions $g(x, t)$ and $\xi(x, t)$ are then obtained by solving Eqs. (8). Finally the boundary-layer equation (7) is solved subject to the boundary conditions (10). The difficulty here is, of course, that one does not know a priori the external velocity distribution for which a solution is being sought.

In the present analysis we use the indirect method of solution. It is assumed that e and d are related by $e+d=(1+d)/2$ and that $h=\xi$. The reason for this choice is quite simple. If this choice is made the second, third and seventh terms in Eq. (7) have coefficients which are of the same form as the corresponding coefficients in the steady-state equivalent of Eq. (7). It is further assumed that u_δ is an explicit function of ξ. This latter assumption yields the additional relation $a/c = d/\xi$.

With $e+d=(1+d)/2$ and $h=\xi$ one obtains from Eqs. (8d–f)

$$g^2 u_\delta = x+p(t), \quad \xi = m(t)[x+p(t)]$$

where $p(t)$ and $m(t)$ are at this point unknown functions of time. If these results are substituted into Eq. (8), Eqs. (8e) and (8f) are satisfied identically, Eqs. (8a) and (8d) both yield

$$(\xi/u_\delta)(du_\delta/d\xi) = d(\xi)$$

and Eqs. (8b) and (8c) yield, respectively,

$$b = (1/u_\delta)(dp/dt) - a$$

$$c = \left(\frac{\xi}{u_\delta}\right)\left[\xi\frac{dm}{dt}\bigg/m^2 + \frac{dp}{dt}\right]$$

Clearly dp/dt and $(1/m^2)(dm/dt)$ must be constants. Taking their values to be K and B, respectively, we have

$$a = \frac{d}{u_\delta}[K+B\xi], \quad b = -a + \frac{K}{u_\delta}$$

$$c = \frac{\xi}{u_\delta}[K+B\xi], \quad d = \frac{\xi}{u_\delta}\frac{du_\delta}{d\xi}$$

$$2e = 1-d, \quad h = \xi$$

It is easily verified by direct substitution that Eqs. (9) are satisfied. Thus the boundary-layer equation (7) becomes

$$f''' + \left(\frac{1+d}{2}\right)ff'' + d(1-f'^2) + \left(\frac{B\xi+K}{u_\delta}\right)d(1-f') + \left[\frac{K}{u_\delta} - \left(\frac{B\xi+K}{u_\delta}\right)d\right]\frac{\eta}{2}f'' - \frac{\xi}{u_\delta}(B\xi+K)\frac{\partial f'}{\partial \xi} + \xi\left\{\frac{\partial f}{\partial \xi}f'' - f'\frac{\partial f'}{\partial \xi}\right\} = 0 \tag{11}$$

where the primes here indicate partial differentiation with respect to η.

Solutions of this type are possible for an external velocity distribution $u_\delta(x,t)$ which may be written explicitly as a function of ξ where $\xi = m(t)[x+p(t)]$. Furthermore since $(1/m^2)(dm/dt) = +B$ and $dp/dt = K$, $m(t)$ and $p(t)$ have the forms $m(t) = 1/(1-Bt)$, $p(t) = Kt$ so that

$$\xi = (x+Kt)/(1-Bt) \tag{12}$$

Discussion of Solutions, the Moving-Wall Case

Consider now the case where K is zero and assume for the moment that we have obtained a solution to a problem in which u_δ is some function of ξ. Further assume that the velocity distribution is one for which separation occurs on the body. In the ξ coordinate system the separation point will be characterized by a single point which we will denote by ξ_0. Then

$$\xi_0 = \frac{x_{sep}}{(1-Bt)}$$

It should be pointed out here that the profile corresponding to ξ_0 represents not one but a sequence of profiles in x, y, t. The separation point then moves along the body with a velocity U_s given by

$$U_s = \frac{dx_{sep}}{dt} = -\xi_0 B \tag{13}$$

so that the separation point moves forward when B is positive and rearward when B is negative. Now let us consider a new coordinate system which is moving with respect to the fixed coordinate system with a velocity $-\xi_0 B$ parallel to the x axis. The moving coordinates, denoted by overbars, are related to the fixed coordinates by

$$x = \bar{x} - B\xi_0 t, \quad \bar{y} = y, \quad \bar{t} = t$$

and the velocities in the moving coordinate system are related to those in the fixed system by

$$u = \bar{u} - B\xi_0, \quad \bar{v} = v$$

It is easily shown that in this moving coordinate system the boundary-layer equations, Eqs. (1) and (2), become

$$\partial\bar{u}/\partial\bar{x} + \partial\bar{v}/\partial\bar{y} = 0 \tag{14}$$

$$\frac{\partial\bar{u}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} = \frac{\partial\bar{u}_\delta}{\partial\bar{t}} + \bar{u}_\delta\frac{\partial\bar{u}_\delta}{\partial\bar{x}} + \nu\frac{\partial^2\bar{u}}{\partial\bar{y}^2} \tag{15}$$

while the boundary conditions are

$$\bar{u}(\bar{x},0,\bar{t}) = B\xi_0, \quad \bar{v}(\bar{x},0,\bar{t}) = 0, \quad \lim_{y\to\infty}\bar{u}(\bar{x},\bar{y},\bar{t}) = \bar{u}_\delta(\bar{x},\bar{t})$$

In the moving coordinate an observer sees the wall moving with a velocity $\xi_0 B$. Since the boundary-layer equations are the same in the moving coordinate system as in the fixed system coordinate system, the analysis given above for semisimilar solutions is applicable also in the moving coordinate system. In the moving system however, the boundary conditions are altered since the x component of velocity is now nonzero at the wall. It is worth noting that there is a certain class of unsteady boundary-layer problems which appear to be steady when observed in the moving coordinate system. These flows have also been studied by the present authors.[11]

There are two advantages to working in the moving coordinate system. First, in this coordinate system the separation point is well defined and easily identifiable. Sears and Telionis[12] have pointed out that the unsteady separation is characterized by a "center of separation," away from the wall, where both the shear and velocity vanish when viewed by an observer moving with the separation point. The moving coordinate system described above is just the coordinate system in which an observer moves with the separation point. Sears and Telionis also point out that this "center of separation" or separation point "exhibits a boundary-layer singularity analogous to Goldstein's." Second, in the moving coordinate system, at least for the case where separation is moving upstream, one never encounters negative velocities. Thus one never is faced with the problem of "integrating upstream" and thereby avoids the possible difficulties which are characteristic of integrating parabolic equations in the wrong direction.

Unsteady, Linearly Retarded Flow

As an example of the application of the above analysis we consider the problem of unsteady linearly retarded flow where the velocity outside the boundary layer is given by

$$u_\delta = U_\infty\left(1 - \frac{Ax}{1-Bt}\right) \tag{16}$$

The steady flow equivalent of this flow ($B=0$) is the classical linearly retarded flow first studied by Howarth.[13] If we take $\xi = x/(1-Bt)$ the abovementioned boundary-layer problem may easily be treated by the method of semisimilar solutions and, as pointed out above, if separation occurs at some point, say ξ_0, the separation point travels with a velocity $-\xi_0 B$. The discussion given above, however, indicates that it is preferable to work in the coordinate system moving with the separation point. In this coordinate system an observer sees an external stream with the velocity

$$\bar{u}_\delta = U_\infty\left[1 - A\frac{(\bar{x} - B\xi_0 t)}{1-Bt}\right] + B\xi_0 \tag{17}$$

This problem is also amenable to treatment with the method of semisimilar solution.

If we now take

$$\xi = \frac{\bar{x} - B\xi_0 t}{1-Bt}, \quad \xi^* = A\xi \quad \text{and} \quad \xi_0{}^* = A\xi_0$$

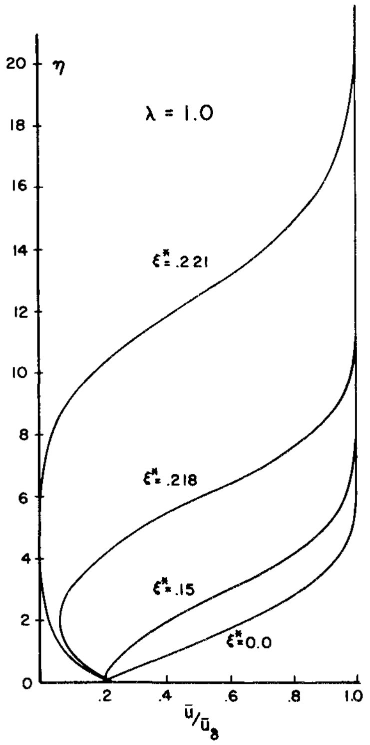

Fig. 1 Velocity profiles in the moving coordinate system for several values of ξ^*.

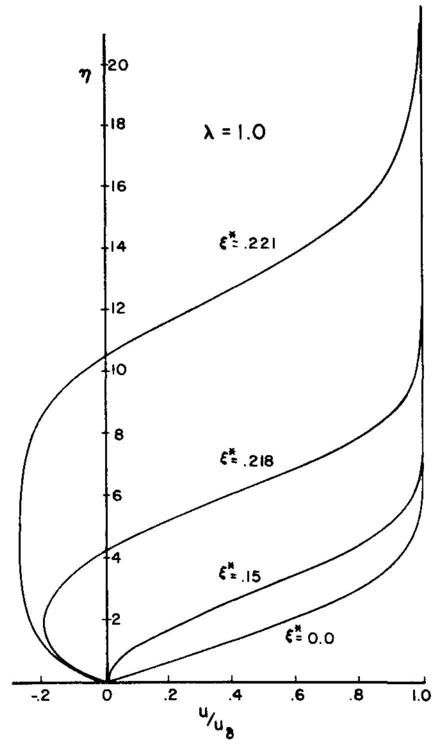

Fig. 2 Velocity profiles in the fixed coordinate system for several values of ξ^*.

Eq. (11) becomes

$$f''' + \frac{1}{2}\left(1 - \frac{\xi^*}{1+\lambda\xi_0^*-\xi^*}\right)ff'' - \left(\frac{\xi^*}{1+\lambda\xi_0^*-\xi^*}\right)(1-f'^2) - \frac{\lambda(\xi^*-\xi_0^*)\xi^*}{[1+\lambda\xi_0^*-\xi^*]^2}(1-f') + \lambda\left\{\frac{\xi^*(\xi^*-\xi_0^*)}{(1+\lambda\xi_0^*-\xi^*)^2} - \frac{\xi_0^*}{(1+\lambda\xi_0^*-\xi^*)}\right\}\frac{\eta}{2}f'' - \left(\frac{\lambda\xi^*(\xi^*-\xi_0^*)}{1+\lambda\xi_0^*-\xi^*}\right)\frac{\partial f'}{\partial \xi^*} + \xi^*\left\{f''\frac{\partial f}{\partial \xi^*} - f'\frac{\partial f'}{\partial \xi^*}\right\} = 0 \quad (18)$$

where $\lambda = B/AU_\infty$ is a measure of the ratio of the unsteady to the steady contributions to the imposed pressure gradient. The boundary conditions which are to be used in the solution of Eq. (18) are

$$f(\xi, 0) = 0, \quad f'(\xi, 0) = \frac{\lambda\xi_0^*}{1+\lambda\xi_0^*-\xi^*}, \quad \lim_{\eta\to\infty} f'(\xi, \eta) = 1 \quad (19)$$

Solutions of Eq. (18), subject to the boundary conditions given by Eqs. (19), are easily obtained using an implicit finite-difference technique similar to that outlined by Blottner.[14] Equation (18) is similar at $\xi^* = 0$, hence the solution procedure is easily started. Since the value of ξ^* at separation, ξ_0^*, is not known at the outset, it is necessary to obtain this value by iteration. For the first iteration ξ_0^* is assumed to be zero. The first iteration results in a new value of ξ_0^*, taken to be the last value of ξ^* for which convergence is obtained for the local velocity profile, which is used in the next iteration. This process is repeated until ξ_0^* does not change in each iteration.

Results have been obtained for values of λ of 0 (Howarth's linearly retarded flow), 0.5 and 1.0. The corresponding values of ξ_0^* are 0.117, 0.161, and 0.221, respectively, corresponding to relative velocities of the separation point, U_s/U_∞, of 0, -0.0805, and -0.221. An attempt was made to find solutions for cases where λ was negative, i.e., cases where the separation is moving rearward. It was found however, that the calculation was unstable in that f' and f'' oscillated as ξ^* increased. This problem deserves further consideration.

Several velocity profiles obtained in the moving coordinate system are shown in Fig. 1. The corresponding velocity profiles, as seen by an observer fixed with respect to the wall, are shown in Fig. 2. In either case profiles are shown for various values of ξ^*. Since ξ^* is a function of both x and t, Figs. 1 and 2 may be interpreted either as velocity profiles seen at a fixed x location for various values of time or as velocity profiles seen at a fixed instant of time at various x locations. As ξ^* approaches its separation value, ξ_0^*, the velocity profiles, in the moving coordinate system, are seen to approach a separation velocity profile characterized by simultaneous vanishing of the shear and the velocity at a point within the boundary layer, thus verifying the Moore-Rott-Sears unsteady boundary-layer separation model. It is interesting to note how very flat the velocity profiles are in this region.

The velocity profile for which the wall shear is zero is one of those presented in Figs. 1 and 2. There was no evidence of any difficulty in integrating the boundary-layer equations in the

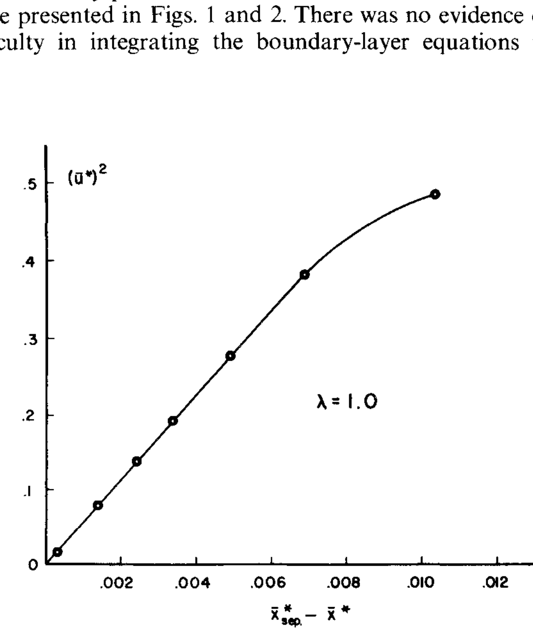

Fig. 3 Variation of the square of the $\bar{x}$ component of velocity, at fixed y and t, as separation is approached.

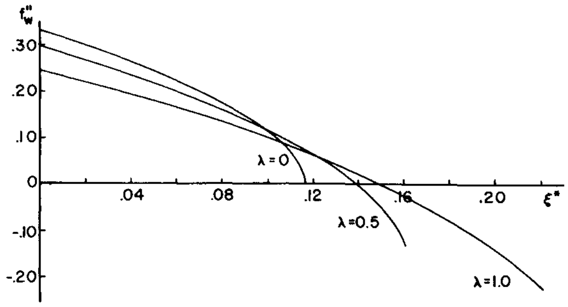

Fig. 4 Variation of wall shear with ξ^*.

vicinity of zero wall shear. On the other hand, Moore[3] and Telionis[6] postulate that there should be a Goldstein-type singularity at the separation point as postulated in the Moore-Rott-Sears model. In the present case, the rapid increase in the number of iterations required to obtain convergence as separation was approached was taken as evidence of such a singularity. To more clearly identify this singularity and determine its nature, the values of the square of the $\bar{x}$ component of velocity, $\bar{u}$, at a fixed value of y and t were obtained and are presented in Fig. 3. The value of y chosen for this determination was the value of y corresponding to the minimum velocity in the last profile, before separation, for which convergence was obtained. From Fig. 3 it is seen that near separation $\bar{u}^2 \sim (\bar{x}_{\text{sep}} - \bar{x})$ indicating a Goldstein square root type singularity at separation and verifying the hypothesis of Moore and Telionis.

The normalized wall shear $f''(\xi^*, 0)$ is shown in Fig. 4 as a function of ξ^*. For the steady-state case ($\lambda = 0$, Howarth's flow) the wall shear has the well-known square-root-type singularity at the point of vanishing wall shear. For the unsteady case the wall shear passes smoothly through the point of zero shear, indicating again that this point has no special significance in unsteady flows. Even as separation is approached there is no evidence of any singular behavior in the wall shear. This one might expect, since for unsteady flow the singular point lies not on the wall but within the boundary layer.

For unsteady flows the proper definition of displacement thickness is somewhat more complicated than that for steady flows.[15] In place of the proper displacement thickness, which is difficult to calculate even for the present simple case, we have calculated the integral Δ^* defined by

$$\Delta^* = \int_0^\infty \left(1 - \frac{u}{u_\delta}\right) d\eta$$

which is the proper definition of the dimensionless displacement thickness for steady flows. This integral, which is taken here as a rough measure of displacement thickness, is presented in Fig. 5 as a function of ξ^* for values of λ of 0.5 and 1.0. Here again if we assume t is constant the curves in Fig. 5 indicate the variation of Δ^* with x at a fixed time. If, on the other hand, we consider a fixed x station, increasing ξ^* in Fig. 5 corresponds to increasing time. Figure 5 indicates then that at a fixed spatial location the boundary layer thickens with time and thickens rapidly as the separation point approaches the location where the observation is being made. The most interesting feature here is that, as one might expect, this "thickness" appears to become unbounded as separation is approached. This result is taken as additional evidence of a singularity in the solution of the boundary layer equations at separation.

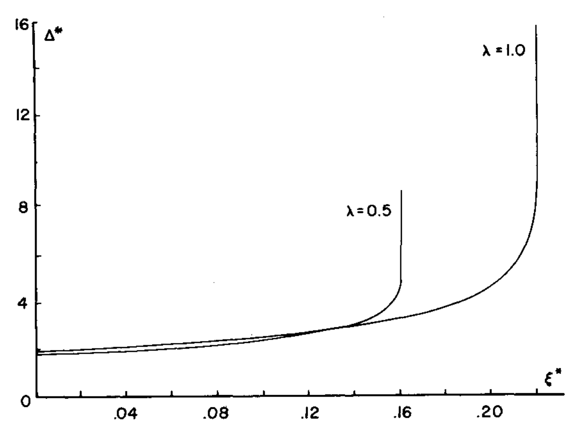

Fig. 5 The variation of the integral Δ^* (pseudo-displacement thickness) with ξ^*.

Conclusions

An analysis is made of a certain class of external flows for which semisimilar solutions of the unsteady laminar boundary-layer equations are possible. It is shown that such solutions may be obtained when the velocity at the edge of the boundary layer is an arbitrary function of ξ where ξ is either $x/(1-Bt)$ or $(x+Kt)/(1-Bt)$. If K is chosen properly the second form of ξ represents the same flow as the first but observed in a coordinate system which moves with the separation point. This feature of the transformation allows one to investigate some unsteady flows where separation occurs in a coordinate system where unsteady separation is most easily identified and analyzed.

As a practical example, an unsteady counterpart of the linearly retarded flow is analyzed. It is shown that in the case where separation is moving forward, separation is characterized by the simultaneous vanishing of the velocity and shear, at a point within the boundary layer, in the coordinate system moving with separation. These results verify the Moore-Rott-Sears model for unsteady boundary-layer separation. Near separation the number of iterations required for convergence in the numerical analysis increased rapidly, a psuedo thickness of the boundary layer appeared to become unbounded and the $\bar{x}$ component of velocity, at fixed values of y and t, varied as the square root of $\bar{x}_s - \bar{x}$. Each of these characteristics is taken as indicative of the singularity in the unsteady boundary-layer equations at separation which was postulated by Moore[3] and Telionis.[6] Finally the authors offer these solutions, or others from this general class, as test cases for verifying three-dimensional (x, y, and t) numerical integration techniques used to investigate unsteady boundary-layer flows.

References

[1] Hall, M. G., "A Numerical Method for Calculating Unsteady Two-Dimensional Laminar Boundary Layers," *Ingenieur-Archiv*, Vol. 38, 1969, pp. 97–106.

[2] Dwyer, H. A., "Calculation of Unsteady and Three-Dimensional Boundary-Layer Flows," *AIAA Journal*, Vol. 11, No. 6, June 1973, pp. 773–774.

[3] Moore, F. K., "On the Separation of the Unsteady Laminar Boundary Layer," in *Boundary Layer Research*, edited by H. Görtler, Springer-Verlag, Berlin, 1958, pp. 296–310.

[4] Rott, N., "Unsteady Flow in the Vicinity of a Stagnation Point," *Quarterly Journal of Applied Mathematics*, Vol. 13, 1956, pp. 444–451.

[5] Sears, W. R., "Some Recent Developments in Airfoil Theory," *Journal of the Aerospace Sciences*, Vol. 23, No. 5, 1956, pp. 490–499.

[6] Telionis, D. P., "Boundary Layer Separation," Ph.D. thesis, Sept. 1970, Cornell Univ., Ithaca, New York.

[7] Telionis, D. P. and Werle, M. J., "Boundary Layer Separation from Moving Boundaries," Rept. VPI-E-72-13, Aug. 1972, Virginia Polytechnic Institute and State University, Blacksburg, Va.

[8] Telionis, D. P., Tsahalis, D. Th., and Werle, M. J., "Numerical Investigation of Unsteady Boundary Layer Separation," *Physics of Fluids*, Vol. 16, No. 7, 1973, pp. 968–973.

[9] Ludwig, G. R., "An Experimental Investigation of Laminar Separation from a Moving Wall," AIAA Paper 64-6, New York, 1964.

[10] Hayasi, N., "On Semi-Similar Solutions of the Unsteady Quasi-Two-Dimensional Incompressible Laminar Boundary Layer Equa-

tions," *Journal of the Physical Society of Japan,* Vol. 17, No. 1, 1962, pp. 194–203.

[11] Williams, J. C., III and Johnson, W. D., "A Note on Unsteady Boundary Layer Separation," *AIAA Journal,* Vol. 12, No. 10, Oct. 1974, pp. 1427-1429.

[12] Sears, W. R. and D. P. Telionis, "Unsteady Boundary Layer Separation," in *Fluid Dynamics of Unsteady, Three-Dimensional and Separated Flows,* edited by F. J. Marshall, Purdue Univ., Lafayette, Ind., 1972, pp. 245–305.

[13] Howarth, L., "On the Solution of the Laminar Boundary Layer Equations," *Proceedings of the Royal Society of London,* Vol. A164, 1938, pp. 547–579.

[14] Blottner, F. G., "Finite-Difference Methods of Solution of the Boundary-Layer Equations," *AIAA Journal,* Vol. 8, No. 2, Feb. 1970, pp. 193–205.

[15] Moore, F. K. and Ostrach, S., "Displacement Thickness of the Unsteady Boundary Layer," *Journal of the Aerospace Sciences,* Vol. 24, No. 1, 1957, pp. 77–78.

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Momentum Integral Method for Viscous-Inviscid Interactions with Arbitrary Wall Cooling

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The momentum integral method of Klineberg is extended to allow the study of two-dimensional laminar viscous-inviscid interactions over a continuous range of wall cooling ratio. By redefining the Klineberg profile quantities, it is shown that the functions appearing in the governing differential equations become, to a good approximation, independent of wall cooling ratio. A considerable simplification of the method is thus affected and flows under nonisothermal wall conditions can be studied. Experimental comparisons are made for a wide class of body shapes which show that the method provides a good description of the major features of viscous-inviscid interactions in both supersonic and hypersonic flow and is superior to other existing methods. In the light of these comparisons, some further improvements to the method are considered.

Nomenclature

a = speed of sound; also Klineberg velocity profile parameter
A = general function
b = enthalpy profile parameter, $-\partial/\partial\eta\,(S/S_w)_{\eta=0}$
B = $(a_e p_e)/(a_\infty p_\infty)$
C = constant in viscosity relationship $(u/u_\infty)/(T/T_\infty)$
C_0-C_6 = coefficients of polynomial functions
C_F = skin-friction coefficient $(\mu\,\partial u/\partial y)_w/(\frac{1}{2}\rho_\infty u_\infty{}^2)$
C_H = heat-transfer coefficient $(k\,\partial T/\partial y)_w/\rho_\infty u_\infty(h_{o_e}-h_{o_w})$
E = $\dfrac{1}{\delta_i^*}\displaystyle\int_o^{\delta_i}\frac{S}{S_w}\,dY$
f = $\dfrac{1}{1+m_e}\left[m_e F+2H-\dfrac{1+2m_e}{m_e}Z\right]-\left[F+\dfrac{Z}{m_e}\right]\dfrac{M_e}{p_e}\dfrac{dp_e}{dM_e}$
F = $H+[(1+m_e)/m_e](1+S_w E)$
h = static enthalpy; also $\dfrac{M_e}{M_\infty}\dfrac{1+m_e}{m_e}\dfrac{Re_{\delta_{i*}}}{C}\left[\dfrac{\tan(\theta_e-\alpha_w)}{1+m_\infty}-\dfrac{\delta_i^*}{B}S_w E\dfrac{d\ln S_w}{dx}\right]$
h_o = total enthalpy
H = $\dfrac{1}{\delta_i^*}\displaystyle\int_o^{\delta_i}\frac{U}{U_e}\left(1-\frac{U}{U_e}\right)dY$
J = $\dfrac{1}{\delta_i^*}\displaystyle\int_o^{\delta_i}\frac{U}{U_e}\left(1-\frac{U^2}{U_e{}^2}\right)dY$
k = thermal conductivity
K = $M_\infty\alpha_i$
m = $[(\gamma-1)/2]M^2$
m_1, m_2 = coefficients of asymptotic expansions [Eq. (14)]
p = static pressure
$p_o, p_1, p_\alpha, p_{\alpha 1}$ = coefficients of asymptotic expansions [Eq. (13)]
P = $\delta_i^*\left[\dfrac{\partial}{\partial Y}\left(\dfrac{U}{U_e}\right)\right]_{Y=0}$
P_1 = $(P-P_B)/(1+0.25S_w)$
Pr = Prandtl number
Q = $-\delta_i^*\left[\dfrac{\partial}{\partial Y}\left(\dfrac{S}{S_w}\right)\right]_{Y=0}$
$\bar{Q}$ = $Q-\dfrac{\delta_i^* Re_{\delta_{i*}}}{BC}\dfrac{M_e}{M_\infty}T^*\dfrac{d\ln S_w}{dx}$
R = $2\delta_i^*\displaystyle\int_o^{\delta_i}\left[\frac{\partial}{\partial Y}\left(\frac{U}{U_e}\right)\right]^2 dY$
Re_x = Reynolds number, $a_\infty M_\infty x/v_\infty$
$Re_{\delta_{i*}}$ = Reynolds number, $a_\infty M_\infty \delta_i^*/v_\infty$
S = total enthalpy function, $[(h_o/h_{o_e})-1]$
T = $\displaystyle\int_o^{(\eta)_{U/U_e=0.99}}\frac{U}{U_e}\frac{S}{S_w}\,d\eta$; also static temperature
T^* = $\dfrac{1}{\delta_i^*}\displaystyle\int_o^{\delta_i}\frac{U}{U_e}\frac{S}{S_w}\,dY$
u, v = velocity components
U, V = Stewartson-Illingworth transformed velocity components
x, y = curvilinear coordinates
X, Y = Stewartson-Illingworth transformed coordinates; also Cartesian coordinates
Z = $\dfrac{1}{\delta_i^*}\displaystyle\int_o^{\delta_i}\frac{U}{U_e}\,dY$

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Index categories: Boundary Layers and Convective Heat Transfer—Laminar; Jets, Wakes, and Viscid-Inviscid Flow Interactions; Supersonic and Hypersonic Flow.

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